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Classical superintegrable $SO(p, q)$ Hamiltonian systems

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Abstract

A family of superintegrable real Hamiltonian systems exhibiting $SO(p, q)$ symmetry is obtained by symmetry reduction from free $SU(p, q)$ integrable Hamiltonian systems. Among them we find Pöschl–Teller potentials. The Hamilton–Jacobi equation is solved in a separable coordinate system in a generic way for the whole family. We also study the projection of the geodesic flow from the complex to the real systems.

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1. Introduction

Completely integrable Hamiltonian systems play a distinguished role among the Hamiltonian dynamical systems. A Hamiltonian system is said to be completely integrable if it has $N - 1$ integrals of motion $Q_j(s, p)$, $j = 1, \ldots, N - 1$, and the set $\{H, Q_j, j = 1, \ldots, N - 1\}$ is functionally independent, well defined in phase space and in involution. When there are more than $N - 1$ integrals of motion (not all of them in involution) the system is called superintegrable. It is maximal superintegrable if this number is equal to $2N - 1$. Superintegrability is also closely related to the fact that the Hamilton–Jacobi (HJ) equation is separable in more than one coordinate system.

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There are not so many examples of superintegrable systems [1], for example, the harmonic oscillator, the Kepler problem, the Calogero–Moser [2,3] and the Smorodinsky–Winternitz [4] systems. They share outstanding properties: at the classical level all the bounded trajectories are closed (periodic) and at the quantum level the energy spectrum is degenerated. Then, it looks very interesting to construct new superintegrable Hamiltonian systems and study their properties. Recently, new systems of this kind have been introduced in [5,6] whose integrals of motion are quadratic functions of the momenta and rational functions of the coordinates. The aim of this paper is to study in a unified way some of these superintegrable systems.

These systems were obtained as follows. We consider a free Hamiltonian system with configuration space the homogeneous space $\mathcal{H}^N = \{y \in \mathbb{C}^{N+1} | g_{\mu\bar{\nu}} y^\mu \bar{y}^\nu = 1\}$ of $SU(p,q)$, $p+q = N+1$, and Hamiltonian

$$H = \tfrac{1}{4} c g^{\bar{\mu}\nu} \bar{p}_\mu p_\nu, \tag{1.1}$$

where g is a Hermitian metric of signature (p,q), c is a positive real constant and the bar stands for complex conjugation.

By means of the Marsden–Weinstein symmetry reduction [7] we obtain a system which is not free, living on a real $SO(p,q)$ homogeneous space $\mathcal{S}^N = \{s \in \mathbb{R}^{N+1} | g_{\mu\nu} s^\mu s^\nu = 1\}$. A potential $V(s)$ appears in the reduced Hamiltonian given by

$$H^r = \tfrac{1}{4} c g^{\mu\nu} p_{s^\mu} p_{s^\nu} + V(s). \tag{1.2}$$

The potential $V(s)$ is a consequence of the reduction procedure as it will be explained later. Note that the Hamiltonian (1.1) has $SU(p,q)$ symmetry while the reduced one (1.2) has $SO(p,q)$ symmetry.

The paper is organized as follows. Section 2 is devoted to describe the geometric structure of the configuration and phase spaces of these systems. The Marsden–Weinstein reduction is carried out in Section 3 for the free $SU(p,q)$ Hamiltonian. In Section 4, the Hamilton–Jacobi equation for the reduced system is solved in a pseudo-spherical coordinate system. Section 5 presents an example in order to enlighten the results of Section 4. Some results about geodesic flows and their projections are presented in Section 6. In Section 7, we state the conclusions and further outlook of this work.

2. Hermitian hyperbolic spaces

The family of Hamiltonian systems which we will study in the sequel is defined in homogeneous spaces of $SU(p,q)$ and $SO(p,q)$. We will describe in this section the geometry of these spaces.

2.1. Homogeneous spaces of $SU(p,q)$ and $SO(p,q)$

In order to present these results in a unified way, we will use the formalism of the orthogonal Cayley–Klein (CK) groups (denoted by $SO_{\hat{\kappa}}(N+1)$, $\hat{\kappa} = (\kappa_1, \ldots, \kappa_N) \in \mathbb{R}^N$), which are the motion groups of the 3^N (N-dimensional) CK geometries [8–10].

Let $\{J_{ab};\ a < b;\ a, b = 0, 1, \ldots, N\}$ be a basis of $so_{\hat\kappa}(N+1)$, the Lie algebra of $SO_{\hat\kappa}(N+1)$. The commutation relations characterizing these algebras are

$$[J_{ab}, J_{ac}] = \kappa_{ab} J_{bc}, \qquad [J_{ac}, J_{bc}] = \kappa_{bc} J_{ab}, \qquad [J_{ab}, J_{bc}] = -J_{ac}, \tag{2.1}$$

where $a < b < c$ and $\kappa_{ab} = \prod_{i=a+1}^{b} \kappa_i$.

The real parameters κ_i can be rescaled to $+1$ or -1 if they are nonzero. We recover the pseudo-orthogonal groups $SO(p, q)$, $p + q = N + 1$, when all the κ_i are nonzero. We will only consider this case in the following.

These groups act linearly on $\mathbb{R}^{N+1}$ by matrix multiplication but the action is not transitive. The pseudo-sphere

$$\mathcal{S}^N_{\hat\kappa} \equiv (x^0)^2 + \kappa_{01}(x^1)^2 + \cdots + \kappa_{0N}(x^N)^2 = 1, \quad x^\mu \in \mathbb{R}, \quad \mu = 0, 1, \ldots, N, \tag{2.2}$$

is the orbit through the point $(1, 0, \ldots, 0)$ and can be considered as the homogeneous space $SO_{\hat\kappa}(N+1)/SO_{\hat\kappa'}(N)$, with $\hat\kappa' = (\kappa_2, \kappa_3, \ldots, \kappa_N)$. In other words, these groups keep invariant the bilinear form defined by

$$g_{\hat\kappa} = \mathrm{diag}(1, \kappa_{01}, \kappa_{02}, \ldots, \kappa_{0N}). \tag{2.3}$$

We will omit the subscript $\hat\kappa$ in the metric in the following.

The $(N+1)$-dimensional matrix realization of the generators J_{ab} is

$$J_{ab} = -\kappa_{ab} E_{ab} + E_{ba}, \quad a < b,\ a, b = 0, 1 \ldots, N,$$

where E_{ab} is the $(N+1) \times (N+1)$ real matrix defined by $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$. A realization in terms of vector fields, associated to the action over $\mathbb{R}^{N+1}$, is

$$J_{ab} = \kappa_{ab} x_b \partial_a - x_a \partial_b.$$

For the pseudo-unitary groups, we can consider in a similar way the family of unitary Cayley–Klein groups, $SU_{\hat\kappa}(N+1)$, $\hat\kappa = (\kappa_1, \ldots, \kappa_N)$ with $\kappa_i \in \mathbb{R}^*$ in our case. These groups act on the complex manifold $\mathbb{C}^{N+1}$ by matrix multiplication keeping invariant the Hermitian form

$$\langle x, y\rangle = g_{\mu\bar\nu} x^\mu \bar y^\nu, \quad x, y \in \mathbb{C}^{N+1}. \tag{2.4}$$

In this case a $(N+1)$-dimensional matrix realization of a basis of $su_{\hat\kappa}(N+1)$ is the following:

$$J_{ab} = -\kappa_{ab} E_{ab} - E_{ba}, \quad K_{ab} = \mathrm{i}(\kappa_{ab} E_{ab} + E_{ba}), \quad a < b,\ a, b = 0, 1 \ldots, N-1,$$
$$H_a = \mathrm{i}(E_{aa} - E_{a+1,a+1}), \quad a = 0, 1 \ldots, N-1.$$

A realization in terms of vector fields is now

$$J_{ab} = \kappa_{ab} y_b \partial_a - y_a \partial_b, \quad K_{ab} = -\mathrm{i}(\kappa_{ab} y_b \partial_a + y_a \partial_b), \quad H_a = \mathrm{i}(-y_a \partial_a + y_{a+1} \partial_{a+1}).$$

Let us denote by $\mathbb{C}_{\hat{\kappa}}^{N+1}$ the complex manifold $\mathbb{C}^{N+1}$ endowed with the Hermitian form defined by the metric g (2.3). The orbit through the point $(1, 0, \ldots, 0) \in \mathbb{C}_{\hat{\kappa}}^{N+1}$ is the real submanifold

$$\mathcal{H}_{\hat{\kappa}}^{N} \equiv |y^0|^2 + \kappa_{01}|y^1|^2 + \cdots + \kappa_{0N}|y^N|^2 = 1, \tag{2.5}$$

which is the homogeneous space $SU_{\hat{\kappa}}(N+1)/SU_{\hat{\kappa}'}(N)$, $\hat{\kappa}' = (\kappa_2, \ldots, \kappa_N)$. If we take into account the action of $U(1)$ on $\mathcal{H}_{\hat{\kappa}}^{N}$, i.e., $y \longrightarrow \mathrm{e}^{\mathrm{i}\alpha} y$, we obtain the Hermitian symmetric spaces

$$C_{\hat{\kappa}}^{N} \equiv \frac{SU_{\hat{\kappa}}(N+1)}{U(1) \times SU_{\hat{\kappa}'}(N)},$$

which are noncompact spaces, except when $\kappa_i = 1$, $i = 1, \ldots, N$, this case corresponding to the complex projective space $CP^N \equiv SU(N+1)/U(N)$. We have a principal fiber bundle with structure group $U(1)$:

$$U(1) \to \mathcal{H}_{\hat{\kappa}}^{N} \to C_{\hat{\kappa}}^{N}.$$

The complex pseudo-spheres $C_{\hat{\kappa}}^{N}$ are the configuration spaces of the Hamiltonians under study.

2.2. *Vector fields on $\mathcal{H}_{\hat{\kappa}}^{N}$ and $C_{\hat{\kappa}}^{N}$*

In the complex manifold $\mathbb{C}_{\hat{\kappa}}^{N+1}$ we consider real vector fields

$$\xi = \xi^{\mu}(y, \bar{y})\partial_{y^\mu} + \bar{\xi}^{\mu}(y, \bar{y})\partial_{\bar{y}^\mu}. \tag{2.6}$$

The condition for a vector field on $\mathbb{C}_{\hat{\kappa}}^{N+1}$ to be a vector field on $\mathcal{H}_{\hat{\kappa}}^{N}$ can be easily expressed through

$$\xi^{\mu}\bar{y}_{\mu} + \bar{\xi}^{\mu} y_{\mu} = 0. \tag{2.7}$$

Owing to the invariance of $\mathcal{H}_{\hat{\kappa}}^{N}$ under the action of the group $SU_{\hat{\kappa}}(N+1)$, the fundamental vector fields of this action are tangent to the hyperboloid. In fact, considering the natural $(N+1) \times (N+1)$ representation of $SU_{\hat{\kappa}}(N+1)$, the fundamental vector fields are

$$\xi_{X_j} = -(X_j)^{\mu}_{\nu} y^{\nu}\partial_{y^\mu} - (\bar{X}_j)^{\mu}_{\nu}\bar{y}^{\nu}\partial_{\bar{y}^\mu}, \tag{2.8}$$

where X_j are the matrices in the corresponding $(N+1) \times (N+1)$ representation of $su_{\hat{\kappa}}(N+1)$, and the vector fields ξ_{X_j}, $j = 1, \ldots, N(N+2)$, are real vector fields in $T\mathcal{H}_{\hat{\kappa}}^{N}$. Relation (2.7) is equivalent to the condition over the matrices X_j

$$X_j K + K X_j^{\dagger} = 0, \tag{2.9}$$

where K is the diagonal matrix corresponding to the Hermitian form g. This relation is the condition for the matrices X_j to be in the Lie algebra $su_{\hat{\kappa}}(N+1)$. The vector field

$$\xi_0 = -y^{\mu}\partial_{y^\mu} - \bar{y}^{\mu}\partial_{\bar{y}^\mu} \tag{2.10}$$

corresponding to the diagonal matrix, which is in $u_{\hat{\kappa}}(N+1)$, is clearly a tangent vector field.

The real vector field on $\mathcal{H}^N_{\hat{\kappa}}$

$$\xi_F = \mathrm{i}y^\mu \partial_{y^\mu} - \mathrm{i}\bar{y}^\mu \partial_{\bar{y}^\mu} \tag{2.11}$$

is tangent to the fibers of $\mathcal{H}^N_{\hat{\kappa}}$ (it is the fundamental vector field associated to the action of $U(1)$ on $\mathcal{H}^N_{\hat{\kappa}}$). According to it, the tangent space to the base manifold $\mathcal{C}^N_{\hat{\kappa}}$ (to the orbit space) in each point $[y]$ is isomorphic to a subspace W_y of the vector space tangent to the hyperboloid. This subspace W_y is orthogonal to the fibers in a point of the hyperboloid projecting on $[y]$: $\pi(y) = [y]$, where π is the natural projection

$$\pi : \mathcal{H}^N_{\hat{\kappa}} \to \mathcal{C}^N_{\hat{\kappa}}. \tag{2.12}$$

Hence, we can write the subspace W_y in terms of the tangent vectors to $\mathcal{H}^N_{\hat{\kappa}}$

$$W_y = \{\xi \in T_y\mathcal{H}^N_{\hat{\kappa}} : \xi^\mu \bar{y}_\mu - \bar{\xi}^\mu y_\mu = 0\}$$

or in terms of the tangent vectors to $\mathcal{C}^N_{\hat{\kappa}}$

$$W_y = \{\xi \in T_{[y]}\mathcal{C}^N_{\hat{\kappa}} : \xi^\mu \bar{y}_\mu = 0\}.$$

The tangent application π_* is an isomorphism when we restrict to the subspace W_y

$$\pi_* : W_y \to T_{[y]}\mathcal{C}^N_{\hat{\kappa}}. \tag{2.13}$$

We can define an almost complex structure in $\mathcal{C}^N_{\hat{\kappa}}$, J' satisfying

$$\pi_* J = J'\pi_*,$$

where J is the complex structure in W_y ($J(y) = \mathrm{i}y$). It can be shown [11] that J' is in fact a complex structure and $\mathcal{C}^N_{\hat{\kappa}}$ a complex manifold.

In each tangent space to $\mathcal{C}^N_{\hat{\kappa}}$ we can define a Hermitian metric with respect to the complex structure J' given by

$$h(\pi_*(\xi), \pi_*(\eta)) = -\frac{2}{c}(g(\xi,\eta) + g(\eta,\xi)), \tag{2.14}$$

where c is a positive constant related to the holomorphic curvature and ξ, η are vectors on W_y.

The action of $U_{\hat{\kappa}}(N+1)$ is transitive on $\mathcal{H}^N_{\hat{\kappa}}$ and $\mathcal{C}^N_{\hat{\kappa}}$. However, it is not effective, and we will use $SU_{\hat{\kappa}}(N+1)$ which has an almost effective action on $\mathcal{C}^N_{\hat{\kappa}}$.

2.3. *Affine coordinates*

In a domain of the hyperboloid $\mathcal{H}^N_{\hat{\kappa}}$ where $y^0 \neq 0$, we can define affine coordinates

$$z^j = \frac{y^j}{y^0}, \quad j = 1, \ldots, N. \tag{2.15}$$

If we consider the orbits on $\mathcal{H}^N_{\hat{\kappa}}$ under the action of the group $U(1)$, we can take a section with the condition $y^0 \in \mathbb{R}^+$. Hence, we can invert Eq. (2.15) and define the homogeneous coordinates y^μ in terms of the affine ones

$$g_{\mu\bar{\nu}} y^\mu \bar{y}^\nu = |y^0|^2 + k_{ij} y^i \bar{y}^j = (y^0)^2(1 + |z|^2) = 1, \tag{2.16}$$

where $k_{ij} = g_{ij}$, $i, j = 1, 2, \ldots, N$ and $|z|^2 = k_{ij} z^i \bar{z}^j$.

Eq. (2.15) implies, due to the choice of the section, that

$$y^0 = \frac{1}{\sqrt{1 + |z|^2}}, \qquad y^j = \frac{z^j}{\sqrt{1 + |z|^2}}, \quad j = 1, \ldots, N. \tag{2.17}$$

We want to compute the Hermitian metric h in the affine coordinates z. To do that we need the expression of the tangent map π_*. Let ξ be a vector field on the subspace W_y of $T_y \mathcal{H}^N_{\hat{\kappa}}$

$$\xi = \xi^\mu \partial_{y^\mu} + \bar{\xi}^\mu \partial_{\bar{y}^\mu}, \quad g_{\mu\bar{\nu}} \xi^\mu \bar{y}^\nu = 0. \tag{2.18}$$

Applying π_*

$$\pi_*(\xi) = \xi^\mu \partial_{y^\mu}(z^j) \partial_{z^j} + \bar{\xi}^\mu \partial_{\bar{y}^\mu}(\bar{z}^j) \partial_{\bar{z}^j}, \tag{2.19}$$

and using

$$\partial_{y^0} z^j = -(1 + |z|^2)^{1/2} z^j, \quad \partial_{y^i} z^j = (1 + |z|^2)^{1/2} \delta^j_i,$$

we get the image under π_*

$$\xi' = \pi_*(\xi) = (1 + |z|^2)^{1/2} (\delta^j_r + k_{rs} z^j \bar{z}^s) \xi^r \partial_{z^j} + \text{c.c.}, \tag{2.20}$$

where c.c. means complex conjugate. Hence

$$\xi'^j = (1 + |z|^2)^{1/2} (\delta^j_r + k_{rs} z^j \bar{z}^s) \xi^r. \tag{2.21}$$

The inverse transformation can be easily computed (remember that π_* is an isomorphism (2.13))

$$\xi^j = (1 + |z|^2)^{-3/2} [(1 + |z|^2) \delta^j_l - k_{lm} z^j \bar{z}^m] \xi'^l, \tag{2.22}$$

and the first component ξ^0 is given by

$$\xi^0 = -(1 + |z|^2)^{-3/2} k_{jl} \bar{z}^l \xi'^j. \tag{2.23}$$

With these expressions we can write explicitly the Hermitian form defined on the tangent space of $\mathcal{C}^N_{\hat{\kappa}}$

$$g_{\mu\bar{\nu}} \xi^\mu \bar{\eta}^\nu = (1 + |z|^2)^{-2} [(1 + |z|^2) k_{jl} - k_{ml} k_{ji} z^m \bar{z}^i] \xi'^j \bar{\eta}'^l, \tag{2.24}$$

which is a generalization of the Fubini–Study metric in a generic noncompact case. Thus, the covariant components of the tensor h are

$$h_{jl} = (1 + |z|^2)^{-2} [(1 + |z|^2) k_{jl} - k_{ml} k_{ji} z^m \bar{z}^i]. \tag{2.25}$$

The contravariant ones are easily obtained. Assuming that

$$h^{jl} = A(k^{jl} + Bz^j\bar{z}^l),$$

and imposing the condition $h^{ij}h_{jl} = \delta^i_l$ we obtain

$$h^{jl} = (1 + |z|^2)(k^{jl} + z^j\bar{z}^l). \tag{2.26}$$

2.4. *Free Hamiltonians in $\mathbb{C}^{N+1}_{\hat{\kappa}}$ and $C^N_{\hat{\kappa}}$*

Let us now consider the free Hamiltonian in $\mathbb{C}^{N+1}_{\hat{\kappa}}$. In the cotangent bundle $T\mathbb{C}^{N+1}_{\hat{\kappa}}$ let us choose the Liouville 1-form

$$\theta = p_\mu\,\mathrm{d}y^\mu + \bar{p}_\mu\,\mathrm{d}\bar{y}^\mu \tag{2.27}$$

in coordinates (y^μ, p_μ). The closed 2-form $\omega = -\,\mathrm{d}\theta$ is given by

$$\omega = \mathrm{d}y^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\bar{y}^\mu \wedge \mathrm{d}\bar{p}_\mu. \tag{2.28}$$

The free Hamiltonian in this space is

$$H = \tfrac{1}{4}cg^{\bar{\mu}\nu}\bar{p}_\mu p_\nu, \tag{2.29}$$

where c is a positive real constant.

The extended action of the group $SU_{\hat{\kappa}}(N+1)$ leaves this Hamiltonian invariant. If we use homogeneous coordinates in $\mathcal{H}^N_{\hat{\kappa}}$ we have to consider the constraint $g_{\bar{\mu}\nu}\bar{y}^\mu y^\nu = 1$. Using affine coordinates in $C^N_{\hat{\kappa}}$ the Hamiltonian can be written as

$$H = h^{ij}\bar{p}_{z^i}p_{z^j} = (1 + k_{lm}z^l\bar{z}^m)(k^{ij}\bar{p}_{z^i}p_{z^j} + (z^i\bar{p}_{z^i})(\bar{z}^j p_{z^j})), \tag{2.30}$$

where p_{z^i} is the momentum conjugate to the coordinate z^i.

3. Symmetry reduction

The use of the Marsden–Weinstein reduction [7] will allow us to construct integrable systems in the manifolds that we have considered above. We will be mainly concerned with the compact Cartan subalgebra of $su_{\hat{\kappa}}(N+1)$, which has dimension N. This procedure is also valid for any maximal abelian subalgebra of dimension N of $su_{\hat{\kappa}}(N+1)$, having a basis formed by purely imaginary matrices [5]. We will use in the sequel a maximal abelian subalgebra, $\mathcal{G}$, of $u_{\hat{\kappa}}(N+1)$ in order to simplify the computations. After reduction we can impose the appropriate conditions to restrict the problem to the real pseudo-sphere. The approach will be made using homogeneous coordinates.

Let $\mathcal{G}^*$ be the dual space of the Lie algebra $\mathcal{G}$. We define the momentum map

$$J : T^*\mathbb{C}^{N+1}_{\hat{\kappa}} \longrightarrow \mathcal{G}^*$$

by

$$\langle J(y,p), X\rangle = F_{\hat{X}^L}(y,p), \quad X \in \mathcal{G}, \tag{3.1}$$

where $\hat{X}^L$ is the fundamental vector field on $T^*\mathbb{C}_{\hat{\kappa}}^{N+1}$ associated to X. We will consider real vector fields on $\mathbb{C}_{\hat{\kappa}}^{N+1}$, $\xi = \xi^\mu \partial_{y^\mu} + \bar{\xi}^\mu \partial_{\bar{y}^\mu}$, their lifted vector fields on $T^*\mathbb{C}_{\hat{\kappa}}^{N+1}$ are given by

$$\xi^L(y,p) = \xi^\mu \partial_{y^\mu} - p_\nu \frac{\partial \xi^\nu}{\partial y^\mu}\partial_{p_\mu} + \text{c.c.} \tag{3.2}$$

Hence, the fundamental vector fields associated to $X \in su_{\hat{\kappa}}(N+1)$ on $\mathbb{C}_{\hat{\kappa}}^{N+1}$ and $T^*\mathbb{C}_{\hat{\kappa}}^{N+1}$ are, respectively,

$$\hat{X} = -(X)^\mu_\nu y^\nu \partial_{y^\mu} + \text{c.c.}, \tag{3.3}$$

$$\hat{X}^L = -(X)^\mu_\nu y^\nu \partial_{y^\mu} + p_\mu (X)^\mu_\nu \partial_{p^\nu} + \text{c.c.} \tag{3.4}$$

On the other hand, the function $F_{\hat{X}^L}$ is defined by the relation $\mathrm{i}(\hat{X}^L)\omega = \mathrm{d}F_{\hat{X}^L}$, with $\omega = \mathrm{d}y^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\bar{y}^\mu \wedge \mathrm{d}\bar{p}_\mu$ the symplectic 2-form on $T^*\mathbb{C}_{\hat{\kappa}}^{N+1}$. Considering an appropriate basis, $\{X_i \mid (X_i)^\mu_\nu = -(\bar{X}_i)^\mu_\nu;\ i = 1, \ldots, \dim \mathcal{G}\}$, of $\mathcal{G}$ we obtain

$$F_{\hat{X}_i^L}(y,p) \equiv F_i(y,p) = -(X_i)^\mu_\nu \left(y^\nu p_\mu - \bar{y}^\nu \bar{p}_\mu\right). \tag{3.5}$$

Let $(r_i) \in \mathcal{G}^*$ be a regular point of the image of the momentum map (3.1). The reduced space is $J^{-1}(r)/\mathcal{G}$ and can be obtained as follows. Let $(s,p) \in T^*\mathbb{C}_{\hat{\kappa}}^{N+1}$ be a point of $J^{-1}(r)$, where $y = s \in \mathbb{R}^{N+1} \subset \mathbb{C}_{\hat{\kappa}}^{N+1}$. Under the action of G, the Lie subgroup of $U_{\hat{\kappa}}(N+1) \equiv SU_{\hat{\kappa}}(N+1) \otimes U(1)$ associated to the Lie algebra $\mathcal{G}$, this point (s,p) is transformed in the following way:

$$y'^\mu \equiv s'^\mu = \Lambda(x)^\mu_\nu s^\nu, \quad p'_\mu = \Lambda^{-1}(x)^\nu_\mu p_\nu, \tag{3.6}$$

with $\Lambda(x) = \exp(x^i X_i) \in G$, $X_i \in \mathcal{G}$ and $x^i \in \mathbb{R}$. Imposing that (s,p) belongs to $J^{-1}(r)$, i.e., $F_i(s,p) = r_i$,

$$-(X_i)^\mu_\nu s^\nu \left(p_\mu - \bar{p}_\mu\right) = r_i,$$

we get a condition over the imaginary part of the momentum p

$$\mathrm{i}(\mathrm{Im}\, p)^{\mathrm{T}} X_i s = -\tfrac{1}{2} r_i. \tag{3.7}$$

Finally, the reduced Hamiltonian is

$$H^r_{\hat{\kappa}} = \tfrac{1}{4} c g^{\bar{\mu}\nu} \bar{p}_\mu p_\nu = \tfrac{1}{4} c (g^{\bar{\mu}\nu}\, \mathrm{Re}\, p_\mu\, \mathrm{Re}\, p_\nu + g^{\bar{\mu}\nu}\, \mathrm{Im}\, p_\mu\, \mathrm{Im}\, p_\nu). \tag{3.8}$$

It is easy to check the $SO_{\hat{\kappa}}(N+1)$ symmetry of the reduced Hamiltonian. If we consider $p_\mu = p_{s^\mu} + \mathrm{i} f^\nu_\mu(s) r_\nu$, where p_{s^μ} is the momentum conjugate to the real coordinate s^μ, the Hamiltonian (3.8) is rewritten as

$$H^r_{\hat{\kappa}} = \tfrac{1}{4} c (g^{\mu\nu} p_{s^\mu} p_{s^\nu} + V(s,r)), \tag{3.9}$$

with $V(s,r) = g^{\mu\nu} f(s)^{\rho}_{\mu} f(s)^{\tau}_{\nu} r_\rho r_\tau$. Note that the group parameters (x^i) are ignorable variables, i.e., they do not appear in the Hamiltonian.

In the case of the compact Cartan subalgebra of $u_{\hat{\kappa}}(N+1)$ we have the basis

$$X_j = \mathrm{i}(E_{j-1,j-1} - E_{jj}), \quad j = 1, \ldots, N, \quad X_0 = \mathrm{diag}(1, \ldots, 1).$$

After reduction by this subalgebra we obtain the potential

$$V = \frac{m_0^2}{(s^0)^2} + \kappa_{01}\frac{m_1^2}{(s^1)^2} + \cdots + \kappa_{0N}\frac{m_N^2}{(s^N)^2}, \tag{3.10}$$

where m_i are some real constants related to r_i. If we are using a subalgebra of $su_{\hat{\kappa}}(N+1)$ instead of $u_{\hat{\kappa}}(N+1)$ we have to impose that $\sum m_i = 0$ and $\mathrm{tr}(x^i X_i) = 0$. On the other hand, if we are in $\mathcal{H}^N_{\hat{\kappa}}$, the point $y = s$ verifies $g_{\mu\nu}s^\mu s^\nu = 1$ and $s^\mu p_{s^\mu} = 0$. Obviously, in this case, the reduced Hamiltonian system is in the real pseudo-sphere $\mathcal{S}^N_{\hat{\kappa}}$.

4. The Hamilton–Jacobi equation

In this section we will solve the Hamilton–Jacobi equation for the Hamiltonian

$$H^r_{\hat{\kappa}} = \frac{c}{4}\left(g^{\mu\nu} p_{s^\mu} p_{s^\nu} + \frac{m_0^2}{(s^0)^2} + \kappa_{01}\frac{m_1^2}{(s^1)^2} + \cdots + \kappa_{0N}\frac{m_N^2}{(s^N)^2}\right). \tag{4.1}$$

Let us consider the following system of "pseudo-spherical" coordinates [10] which allow us to parametrize the pseudo-sphere $\mathcal{S}^N_{\hat{\kappa}}$:

$$\begin{aligned}
s^0 &= C_{\kappa_{0N}}(\phi_N) C_{\kappa_{0,N-1}}(\phi_{N-1}) \ldots C_{\kappa_{01}}(\phi_1),\\
s^1 &= C_{\kappa_{0N}}(\phi_N) C_{\kappa_{0,N-1}}(\phi_{N-1}) \ldots S_{\kappa_{01}}(\phi_1),\\
s^2 &= C_{\kappa_{0N}}(\phi_N) C_{\kappa_{0,N-1}}(\phi_{N-1}) \ldots S_{\kappa_{02}}(\phi_2),\\
s^3 &= C_{\kappa_{0N}}(\phi_N) C_{\kappa_{0,N-1}}(\phi_{N-1}) \ldots S_{\kappa_{03}}(\phi_3),\\
&\vdots\\
s^{N-1} &= C_{\kappa_{0N}}(\phi_N) S_{\kappa_{0,N-1}}(\phi_{N-1}),\\
s^N &= S_{\kappa_{0N}}(\phi_N).
\end{aligned} \tag{4.2}$$

The symbols $C_\kappa(x)$ and $S_\kappa(x)$ stand for a generalized version of the trigonometric functions $\cos x$ and $\sin x$, respectively. They are defined in the following way [8]:

$$C_\kappa(x) = \begin{cases} \cos\sqrt{\kappa}x, & \kappa > 0,\\ 1, & \kappa = 0,\\ \cosh\sqrt{-\kappa}x, & \kappa < 0,\end{cases} \qquad S_\kappa(x) = \begin{cases} \sin(\sqrt{\kappa}x)/\sqrt{\kappa}, & \kappa > 0,\\ x, & \kappa = 0,\\ \sinh(\sqrt{-\kappa}x)/\sqrt{-\kappa}, & \kappa < 0.\end{cases}$$

When $\kappa = 1$ we recover the usual trigonometric functions, when $\kappa = -1$ we find the hyperbolic functions and when $\kappa = 0$ the parabolic or galilean functions, $C_0(x) = 1$ and $S_0(x) = x$. These generalized functions verify, for instance, the following properties:

$$C_\kappa^2(x) + \kappa S_\kappa^2(x) = 1, \quad \frac{\mathrm{d}}{\mathrm{d}x}C_\kappa(x) = -\kappa S_\kappa(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}S_\kappa(x) = C_\kappa(x).$$

In such coordinates (4.2) the Hamiltonian (4.1) is rewritten as

$$H_{\hat{\kappa}}^{r} = \kappa_{0N} p_{\phi_N}^2 + \frac{1}{C_{\kappa_{0N}}^2(\phi_N)} \Bigg[\kappa_{0,N-1} p_{\phi_{N-1}}^2 + \frac{1}{C_{\kappa_{0,N-1}}^2(\phi_{N-1})} \times \Bigg[\dots \Bigg[\kappa_{01} p_{\phi_1}^2 + \frac{m_0^2}{C_{\kappa_{01}}^2(\phi_1)} + \frac{\kappa_{01} m_1^2}{S_{\kappa_{01}}^2(\phi_1)} \Bigg] \dots \Bigg] + \frac{\kappa_{0,N-1} m_{N-1}^2}{S_{\kappa_{0,N-1}}^2(\phi_{N-1})} \Bigg] + \frac{\kappa_{0N} m_N^2}{S_{\kappa_{0N}}^2(\phi_N)}. \quad (4.3)$$

The HJ equation associated to this Hamiltonian (4.3) is obtained after the substitution $p_{\phi_i} \longrightarrow \partial S/\partial \phi_i$ in $H_{\hat{\kappa}}^r$. So, we get $H_{\hat{\kappa}}^r(\partial S/\partial \phi_i, \phi_i) = E$. This equation is completely separable, and has a solution of the form

$$S(\phi_1, \dots, \phi_N) = \sum_{i=1}^{N} S_i(\phi_i) - Et. \quad (4.4)$$

We can separate this equation in the set of equations

$$\begin{aligned} \kappa_{0N} \left(\frac{\partial S_N}{\partial \phi_N} \right)^2 + \frac{\alpha_{N-1}}{C_{\kappa_{0N}}^2(\phi_N)} + \frac{\kappa_{0N} m_N^2}{S_{\kappa_{0N}}^2(\phi_N)} &= \alpha_N, \\ \kappa_{0N-1} \left(\frac{\partial S_{N-1}}{\partial \phi_{N-1}} \right)^2 + \frac{\alpha_{N-2}}{C_{\kappa_{0,N-1}}^2(\phi_{N-1})} + \frac{\kappa_{0,N-1} m_{N-1}^2}{S_{\kappa_{0,N-1}}^2(\phi_{N-1})} &= \alpha_{N-1}, \\ &\vdots \\ \kappa_{01} \left(\frac{\partial S_1}{\partial \phi_1} \right)^2 + \frac{\alpha_0}{C_{\kappa_{01}}^2(\phi_1)} + \frac{\kappa_{01} m_1^2}{S_{\kappa_{01}}^2(\phi_1)} &= \alpha_1, \end{aligned} \quad (4.5)$$

with $\alpha_N = E$ and $\alpha_0 = m_0^2$. Hence, the general solution of (4.5) is

$$S_i = \pm \int \left[\kappa_{0i} \left(\alpha_i - \frac{\alpha_{i-1}}{C_{\kappa_{0i}}^2(\phi_i)} - \frac{\kappa_{0i} m_i^2}{S_{\kappa_{0i}}^2(\phi_i)} \right) \right]^{1/2} \mathrm{d}\phi_i, \quad i = 1, 2, \dots, N. \quad (4.6)$$

If we consider the separation constants as the new momenta, the conjugate coordinates will now be

$$\beta_i = \frac{\partial S}{\partial \alpha_i} = \frac{\partial S_i}{\partial \alpha_i} + \frac{\partial S_{i+1}}{\partial \alpha_i}, \quad i = 1, \dots, N-1, \quad (4.7)$$

$$\beta_N = \frac{\partial S}{\partial \alpha_N} = \frac{\partial S_N}{\partial \alpha_N} - t, \quad (4.8)$$

which can be solved by an iterative procedure as follows.

From (4.6), for $i = N$, we get

$$\frac{\partial S_N}{\partial \alpha_N} = \pm \frac{\kappa_{0N}}{2} \int \left[\kappa_{0N} \left(\alpha_N - \frac{\alpha_{N-1}}{C_{\kappa_{0N}}^2(\phi_N)} - \frac{\kappa_{0N} m_N^2}{S_{\kappa_{0N}}^2(\phi_N)} \right) \right]^{-1/2} \mathrm{d}\phi_N. \quad (4.9)$$

Using the change of variable $u_N = C^2_{\kappa_{0N}}(\phi_N)$, (4.9) can be written as

$$\frac{\partial S_N}{\partial \alpha_N} = \mp\frac{1}{4}\int [-\alpha_N u_N^2 + b_N u_N - \alpha_{N-1}]^{-1/2}\, \mathrm{d}u_N, \tag{4.10}$$

where $b_N = \alpha_N + \alpha_{N-1} - m_N^2$. The integral (4.10) depends on the sign of α_N, that is, the sign of the energy. We can distinguish the following cases:

(1) $\alpha_N > 0$

$$\frac{\partial S_N}{\partial \alpha_N} = \pm\frac{1}{4\sqrt{\alpha_N}} \sin^{-1}\left(\frac{-2\alpha_N u_N + b_N}{\sqrt{b_N^2 - 4\alpha_N\alpha_{N-1}}}\right). \tag{4.11}$$

(2) $\alpha_N < 0$

(a) $b_N^2 - 4\alpha_N\alpha_{N-1} < 0$

$$\frac{\partial S_N}{\partial \alpha_N} = \mp\frac{1}{4\sqrt{-\alpha_N}} \sinh^{-1}\left(\frac{-2\alpha_N u_N + b_N}{\sqrt{-b_N^2 + 4\alpha_N\alpha_{N-1}}}\right). \tag{4.12}$$

(b) $b_N^2 - 4\alpha_N\alpha_{N-1} \geq 0$

$$\frac{\partial S_N}{\partial \alpha_N} = \mp\frac{1}{4\sqrt{-\alpha_N}} \times \log(-2\sqrt{-\alpha_N(-\alpha_N u_N^2 + b_N u_N - \alpha_{N-1})} - 2\alpha_N u_N + b_N). \tag{4.13}$$

(3) $\alpha_N = 0$

$$\frac{\partial S_N}{\partial \alpha_N} = \mp\frac{1}{2}\frac{\sqrt{(\alpha_{N-1} - m_N^2)u_N - \alpha_{N-1}}}{\alpha_{N-1} - m_N^2}. \tag{4.14}$$

From (4.4) and (4.8) we obtain

$$t - t_0 = \frac{\partial S_N}{\partial \alpha_N}, \tag{4.15}$$

where $t_0 = -\beta_N$, and the solutions for the u_N variable are

(1) $$u_N = \frac{b_N \mp \sqrt{b_N^2 - 4\alpha_N\alpha_{N-1}}\sin(4\sqrt{\alpha_N}(t - t_0))}{2\alpha_N}. \tag{4.16}$$

(2)(a) $$u_N = \frac{b_N \mp \sqrt{-b_N^2 + 4\alpha_N\alpha_{N-1}}\sinh(4\sqrt{-\alpha_N}(t - t_0))}{2\alpha_N}. \tag{4.17}$$

(b) $$u_N = \frac{1}{4\alpha_N}(2b_N + (4\alpha_N\alpha_{N-1} - b_N^2)\exp(\mp 4\sqrt{-\alpha_N}(t - t_0)) - \exp(\pm 4\sqrt{-\alpha_N}(t - t_0))). \tag{4.18}$$

(3) $$u_N = 4(\alpha_{N-1} - m_N^2)(t - t_0)^2 + \frac{\alpha_{N-1}}{\alpha_{N-1} - m_N^2}. \tag{4.19}$$

The remaining coordinates β_i, $i = 1, \dots, N-1$, can be computed in a similar way, taking into account (4.7). The first term in the RHS of this expression is obtained as in the case $\partial S_N/\partial\alpha_N$. It is enough to change the index N by i in expressions (4.9)–(4.19). The computation of the second term is as follows. From (4.6) we get

$$\begin{aligned}\frac{\partial S_{i+1}}{\partial\alpha_i} &= \mp\frac{\kappa_{0,i+1}}{2}\int C_{\kappa_{0,i+1}}^{-2}(\phi_{i+1}) \\ &\quad\times\left[\kappa_{0,i+1}\left(\alpha_{i+1} - \frac{\alpha_i}{C_{\kappa_{0,i+1}}^2(\phi_{i+1})} - \frac{\kappa_{0,i+1}m_{i+1}^2}{S_{\kappa_{0,i+1}}^2(\phi_{i+1})}\right)\right]^{-1/2} \mathrm{d}\phi_{i+1},\end{aligned} \tag{4.20}$$

Using again the change of variable $u_{i+1} = C_{\kappa_{0,i+1}}^2(\phi_{i+1})$, (4.20) is written as

$$\frac{\partial S_{i+1}}{\partial\alpha_i} = \pm\frac{1}{4}\int u_{i+1}^{-1}[-\alpha_{i+1}u_{i+1}^2 + b_{i+1}u_{i+1} - \alpha_i]^{-1/2}\,\mathrm{d}u_{i+1}, \tag{4.21}$$

where $b_{i+1} = \alpha_{i+1} + \alpha_i - m_{i+1}^2$.

The integrals (4.21) also depend on the sign of α_i. The results can be listed as follows:

(1) $\alpha_i > 0$

$$\frac{\partial S_{i+1}}{\partial\alpha_i} = \pm\frac{1}{4\sqrt{\alpha_i}}\sin^{-1}\left(\frac{b_{i+1}u_{i+1} - 2\alpha_i}{u_{i+1}\sqrt{b_{i+1}^2 - 4\alpha_{i+1}\alpha_i}}\right). \tag{4.22}$$

(2) $\alpha_i < 0$

(a) $b_{i+1}^2 - 4\alpha_{i+1}\alpha_i < 0$

$$\frac{\partial S_{i+1}}{\partial\alpha_i} = \pm\frac{1}{4\sqrt{-\alpha_i}}\sinh^{-1}\left(\frac{b_{i+1}u_{i+1} - 2\alpha_i}{u_{i+1}\sqrt{-b_{i+1}^2 + 4\alpha_{i+1}\alpha_i}}\right). \tag{4.23}$$

(b) $b_{i+1}^2 - 4\alpha_{i+1}\alpha_i \ge 0$

$$\begin{aligned}\frac{\partial S_{i+1}}{\partial\alpha_i} &= \mp\frac{1}{4\sqrt{-\alpha_i}} \\ &\quad\times\log\left[\frac{2\sqrt{-\alpha_i(-\alpha_{i+1}u_{i+1}^2 + b_{i+1}u_{i+1} - \alpha_i)}}{u_{i+1}} - \frac{2\alpha_i}{u_{i+1}} + b_{i+1}\right].\end{aligned} \tag{4.24}$$

(3) $\alpha_i = 0$

$$\frac{\partial S_{i+1}}{\partial\alpha_i} = \mp\frac{\sqrt{-\alpha_{i+1}u_{i+1}^2 + (\alpha_{i+1} - m_{i+1}^2)u_{i+1}}}{2u_{i+1}(\alpha_{i+1} - m_{i+1}^2)}. \tag{4.25}$$

Expressions (4.22)–(4.25) added to expressions (4.11)–(4.14) (substituting the index N by i) give the complete solution of β_i, $i = 1, \dots, N-1$ (according to the sign of α_i).

As an example, let us consider the case $\kappa_i = 1$, $i = 1, \dots, N$. The manifold is now the real sphere S^N and the angles ϕ_i in (2.8) take the values $0 \le \phi_1 < 2\pi$ and $-\frac{1}{2}\pi < \phi_j \le \frac{1}{2}\pi$, $j \neq 1$. From (2.9), all the constants α_i are positive. Hence, the solutions are

$$\cos^2(\phi_N) = \frac{b_N \mp \sqrt{b_N^2 - 4\alpha_N\alpha_{N-1}}\,\sin(4\sqrt{\alpha_N}(t-t_0))}{2\alpha_N}, \tag{4.26}$$

$$\begin{aligned}\cos^2(\phi_j) = \frac{1}{2\alpha_j}\Bigg[b_j - \frac{1}{\cos^2(\phi_{j+1})}\left[\frac{b_j^2 - 4\alpha_j\alpha_{j-1}}{b_{j+1}^2 - 4\alpha_{j+1}\alpha_j}\right]^{1/2} \\ \times [2\sqrt{\alpha_j}[(b_{j+1} - \alpha_{j+1}\cos^2(\phi_{j+1}))\cos^2(\phi_{j+1})]^{1/2}\sin(\pm 4\beta_j\sqrt{\alpha_j}) \\ \pm (b_{j+1}\cos^2(\phi_{j+1}) - 2\alpha_j)\cos(4\beta_j\sqrt{\alpha_j})]\Bigg], \quad i = 1, \dots, N-1.\end{aligned} \tag{4.27}$$

5. Example

As an application of the theory developed in Section 4, we will study here a particular case.

Let us consider a system associated to $SU(2,1)$. This group belongs to the CK family and appears, for instance, when $N = 2$ and $\hat{\kappa} = (\kappa_1 = 1, \kappa_2 = -1)$. Note that from a geometric point of view, the (reduced) real hyperboloid is endowed with a co-hyperbolic geometry [8]. According to expression (4.3) we can write the Hamiltonian of our system as

$$H_{\hat{\kappa}}^r = -p_{\phi_2}^2 + \frac{1}{\cosh^2\phi_2}\left[p_{\phi_1}^2 + \frac{m_0^2}{\cos^2\phi_1} + \frac{m_1^2}{\sin^2\phi_1}\right] - \frac{m_2^2}{\sinh^2\phi_2}. \tag{5.1}$$

Taking the values $m_0 = 1$, $m_1 = -0.5$, $m_2 = -0.5$ we obtain the numerical values of the constants

$$\alpha_1 = 3.4841, \quad \alpha_2 = E = -0.3281. \tag{5.2}$$

We choose as initial conditions for our physical system the following ones:

$$\phi_1(0) = 0.78, \quad \phi_2(0) = 3, \quad p_{\phi_1}(0) = 1, \quad p_{\phi_2}(0) = -0.6. \tag{5.3}$$

Using the property of separability of the system we obtain (4.6)

$$\frac{\partial S_2}{\partial \alpha_2} = \mp\frac{1}{2}\int\left[\left(-\alpha_2 + \frac{\alpha_1}{\cosh^2\phi_2} - \frac{m_2^2}{\sinh^2\phi_2}\right)\right]^{-1/2} \mathrm{d}\phi_2. \tag{5.4}$$

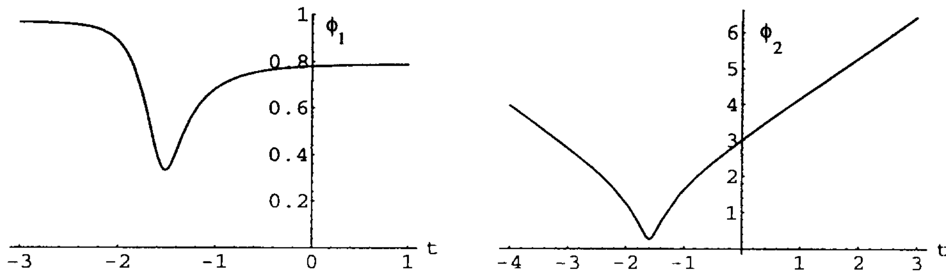

Fig. 1. Time evolution of ϕ_1 and ϕ_2.

Making the change of variable $u_2 = \cosh^2 \phi_2$, we finally arrive at

$$u_2 = \frac{1}{4\alpha_2}(2b_2 + (4\alpha_2\alpha_1 - b_2^2)\exp(4\sqrt{-\alpha_2}(t - t_0)) - \exp(-4\sqrt{-\alpha_2}(t - t_0))). \tag{5.5}$$

Since $\partial S/\partial\alpha_1 = \partial S_2/\partial\alpha_1 + \partial S_1/\partial\alpha_1$ we must evaluate both terms, i.e.,

$$\frac{\partial S_2}{\partial \alpha_1} = \pm\frac{1}{2}\int \cosh^{-2}(\phi_2)\left[\left(-\alpha_2 + \frac{\alpha_1}{\cosh^2\phi_2} - \frac{m_2^2}{\sinh^2\phi_2}\right)\right]^{-1/2} \mathrm{d}\phi_2, \tag{5.6}$$

and

$$\frac{\partial S_1}{\partial \alpha_1} = \pm\frac{1}{2}\int \left[\left(\alpha_1 - \frac{\alpha_0}{\cos^2\phi_1} - \frac{m_1^2}{\sin^2\phi_1}\right)\right]^{-1/2} \mathrm{d}\phi_1. \tag{5.7}$$

Taking into account the changes of variable $u_1 = \cos^2\phi_1$, $u_2 = \cosh^2\phi_2$, and performing the integration (note that in this case $\alpha_1 > 0$) we get

$$u_1 = \frac{b_1 \mp \sqrt{b_1^2 - 4\alpha_1\alpha_0}\sin(4\sqrt{\alpha_1}(\beta_1 - \partial S_2/\partial\alpha_1))}{2\alpha_1}. \tag{5.8}$$

The time evolution of the variables $\phi_1 = \cos^{-1}(\sqrt{u_1})$ and $\phi_2 = \cosh^{-1}(\sqrt{u_2})$ is given in the graphics (Fig. 1).

6. Projection of geodesic flows

The geodesic flow in $C_{\hat{\kappa}}^N$ is easily obtained from the Hamilton equations and the flow in the real pseudo-sphere $\mathcal{S}_{\hat{\kappa}}^N$ is achieved from the first one by projection.

The equations of motion in $\mathcal{H}_{\hat{\kappa}}^N$ can be computed using the method of Lagrange multipliers. The new Hamiltonian H_λ takes into account the condition $g_{\mu\bar{\nu}}y^\mu\bar{y}^\nu = 1$, i.e., the Hamiltonian system is in $C_{\hat{\kappa}}^N$, so

$$H_\lambda = \tfrac{1}{4}cg^{\bar{\mu}\nu}\bar{p}_\mu p_\nu + \lambda(g_{\mu\bar{\nu}}y^\mu\bar{y}^\nu - 1),$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier to be determined. The Hamilton equations give rise to the equation

$$\ddot{y}^{\mu} + \tfrac{1}{4}\lambda c y^{\mu} = 0. \tag{6.1}$$

There are three cases, according to the values of λ: positive, negative or zero. However, we shall jointly handle all of them. The solution of (6.1) is

$$y^{\mu}(t) = A^{\mu}C_{\omega}(t) + B^{\mu}S_{\omega}(t), \tag{6.2}$$

$$p_{\mu}(t) = -\omega A_{\mu}S_{\omega}(t) + B_{\mu}C_{\omega}(t), \tag{6.3}$$

where $\omega = \frac{1}{4}\lambda c$, and A^{μ}, B^{μ} are integration constants satisfying $g_{\mu\bar{\nu}}A^{\mu}\bar{A}^{\nu} = 1$, $g_{\mu\bar{\nu}}B^{\mu}\bar{B}^{\nu} = \omega$, and $g_{\mu\bar{\nu}}(A^{\mu}\bar{B}^{\nu} + \bar{A}^{\mu}B^{\nu}) = 0$. The geodesic flow on $\mathcal{H}^N_{\hat{\kappa}}$ is given by (6.2). Considering affine coordinates it is easy to obtain the flow on $\mathcal{C}^N_{\hat{\kappa}}$.

The Lagrange parameter λ is, as usual, the energy of the system $E = \frac{1}{4}cg^{\bar{\mu}\nu}\bar{p}_{\mu}p_{\nu} = \lambda$. Note that when $E > 0$ the motion is bounded and when $E \leq 0$ the motion is unbounded.

Taking into account that we have made the symmetry reduction using the compact Cartan subalgebra of $su_{\hat{\kappa}}(N+1)$ one easily gets the projected flow on $\mathcal{S}^N_{\hat{\kappa}}$. So, a generic element of the Lie group associated to this subalgebra is given by $\Lambda(x) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}x^0}, \mathrm{e}^{\mathrm{i}x^1}, \ldots, \mathrm{e}^{\mathrm{i}x^N})$, with $\sum_{i=0}^{N} x^i = 0$. Hence, $y^{\mu} = \Lambda(x)^{\mu}_{\nu}s^{\nu} = \mathrm{e}^{\mathrm{i}x^{\mu}}s^{\mu}$ (no sum in the index μ). The flow can be projected in the following way:

$$(s^{\mu})^2 = |y^{\mu}|^2 = |A^{\mu}|^2C^2_{\lambda}(t) + |B^{\mu}|^2S^2_{\lambda}(t) + 2\,\mathrm{Re}\,(A^{\mu}\bar{B}^{\mu})C_{\lambda}(t)S_{\lambda}(t),$$

with the conditions $g_{\mu\bar{\nu}}A^{\mu}\bar{A}^{\nu} = 1$, $g_{\mu\bar{\nu}}B^{\mu}\bar{B}^{\nu} = \lambda$ and $\mathrm{Re}\,(g_{\mu\bar{\nu}}A^{\mu}\bar{B}^{\nu}) = 0$.

This solution was obtained in Section 4 using pseudo-spherical coordinates.

7. Concluding remarks

This paper presents a realization of the conjecture given in [12] in the sense that any completely integrable system "should arise as reduction of a simple one". In our case these simple systems are free systems, which are geodesic flows in a homogeneous space of a Lie group.

We can consider another maximal abelian subalgebras (MASA) of $su(p,q)$ [14] instead of the Cartan subalgebras. The method is similar and some results can be found in [5,6]. For low dimensions some of our potentials can be considered as generalizations of the celebrated Pösch–Teller [13] and Morse (for nilpotent MASA) potentials [15] which have many applications in atomic physics. See also the work [16] for other $SO(p,q)$-invariant integrable systems.

A well-known Hamiltonian very closely related with our systems is the Rosochatius Hamiltonian

$$H = \frac{1}{2}\sum_{\mu=0}^{N} p_{\mu}^2 + \sum_{\mu=0}^{N}\frac{a_{\mu}^2}{x_{\mu}^2} + \sum_{\mu=0}^{N} b_{\mu}^2 x_{\mu}^2,$$

where $\sum_{\mu=0}^{N} x_\mu^2 = 1$ and a_μ and b_μ are constants [17–20]. If one does not consider the harmonic terms of this potential, we have the reduced Hamiltonian for CP^N [21] to the real sphere S^N. As we have seen, our Hamiltonians are a generalization of it. It is interesting to note that this compact case, i.e., when $\kappa_i = 1$ for all i, has been studied by Gagnon [22] and his results are in agreement with ours (see also [21]). As it was remarked in [20], there is a deep connection between the geometric and algebraic structures of many Hamiltonian systems.

It is worthy to note that our method, besides to handle in a unified way a big family of these systems (containing both compact and noncompact cases), enlightens the connection between both structures.

Further work on these systems, like reduction using another type of abelian subalgebras and quantum versions is in progress.

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